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# Nonlinear Approach to Electrodynamics

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A nonlinear approach to electrodynamics is reviewed. On imposing a nonlinear constraint $A_\mu A^\mu = -\rho^2$, together with the usual gauge-invariant electromagnetic field Lagrangian, it is found that the resulting equations of motion have, besides the photon, a static spherically symmetric extended solution which may be regarded as a charged particle. A magnetic dipole moment (spin) can also arise as a solution of the equations of motion if, as expected, it is treated as a first-order quantum effect. In the limit for "small" quantum fields and pointlike charged particles, the quantum mechanical equivalence of the approach with the usual Lagrangian formulation of the electromagnetic interaction of a charged scalar field is heuristically shown. Moreover the possibility of constructing charged fermion fields from the solution having both a charge and a magnetic moment is illustrated. In such an approach the photon is associated with the spontaneous breaking of Lorentz symmetry, and the emission of soft photons does not exhibit any infrared divergences.

## 1. INTRODUCTION

In order to remove the difficulties associated with point charge electrons, several years ago Dirac suggested that electrodynamics should be built from the classical theory of the motion of a continuous stream of electricity rather than the motion of point charges (Dirac, 1951; 1952; 1954).[1] The approach was based on the observation that gauge invariance reflects the fact that the usual electromagnetic field theory involves more dynamical variables than are physically necessary. Therefore a suitably chosen constraint for the electromagnetic field $A_\mu$ may be used to destroy gauge invariance and make the superfluous variables acquire physical

[1]We observe that Dirac initially took the electromagnetic field as proportional to the velocity four-vector of the electric stream and never considered particlelike solutions to his equations.

significance and describe charged currents. In particular the constraint consisted in requiring the electromagnetic field to satisfy[2]

$$A_\mu^2 = -\rho^2 \tag{1}$$

with $\rho$ real.

The above choice of constraint for the electromagnetic field can be motivated by considering the classical limit for a Lagrangian density $\mathfrak{L}'$ describing a local charged scalar field $\varphi(x)$ and the electromagnetic field together with their interaction. Indeed on taking

$$\mathfrak{L}' = -\frac{1}{4}F_{\mu\nu}^2 - \left(\partial_\mu\varphi^\dagger - \frac{ie}{\hbar}A_\mu\varphi^\dagger\right)\left(\partial_\mu\varphi + \frac{ie}{\hbar}A_\mu\varphi\right) - \frac{m^2}{\hbar^2}\varphi^\dagger\varphi \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we can obtain the classical limit of $\mathfrak{L}'$ by setting

$$\varphi = Fe^{is/\hbar} = \left[\sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n F_n\right]e^{is/\hbar} \tag{3}$$

where $S$ is related to the mechanical action and may in turn be expanded as

$$S = \sum_{n=0}^{\infty}\left(\frac{\hbar}{i}\right)^n S_n \tag{4}$$

To lowest order in $\hbar$ Equation (2) becomes

$$\mathfrak{L}'_{\text{class.}} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{F_0^2}{\hbar^2}\left[\left(\partial_\mu S_0 + eA_\mu\right)^2 + m^2\right] \tag{5}$$

and on setting the coefficient of $F_0^2$ in the above equal to zero

$$\left(\frac{1}{e}\partial_\mu S_0 + A_\mu\right)^2 = -\frac{m^2}{e^2} \tag{6}$$

we are just imposing the Hamilton–Jacobi equation for a charged particle moving according to the Lorentz equations of motion in the field of the potential $A_\mu$. Further we observe that Equation (6) after making the gauge

[2] Our metric is $\delta_{\mu\nu}$: $\mu, \nu = 1$–$4$ and $x_4 = ix_0$. Greek indices run from 1 to 4 and Latin indices from 1 to 3. Repeated indices are summed over and we shall use units for which $\hbar = c = 1$ except in Sections 1 and 2, where we do not set $\hbar = 1$.

transformation

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu S_0$$

just becomes the constraint Equation (1) with the identification $\rho^2 = m^2/e^2$. Naturally this is possible to the extent that there exists a solution to the classical equation of motion Equation (6).

At this point we note that instead of Equation (2) we are just left with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 \tag{7}$$

with the constraint Equation (1) for the electromagnetic field, the charged fields having disappeared from the Lagrangian density. Clearly we have then achieved the aim of describing the charged particles in terms of the electromagnetic field itself, that is as singularities in the field as we shall later see.

All our considerations will then begin from the "free" electromagnetic Lagrangian density Equation (7) where the electromagnetic field satisfies the constraint Equation (1). A further support for the choice $\rho^2 > 0$ can also be given by examining the propagation of the solutions to the resulting equations of motion. Indeed on computing the normals to the characteristic surfaces one can verify that $\rho^2 < 0$ leads to acausal propagation. This is not surprising since according to the above heuristic procedure ($\rho^2 = m^2/e^2$), $\rho^2 < 0$ would correspond to an imaginary mass and therefore a tachyon.

In the next section we shall briefly review the solutions to the nonlinear equations of motion obtained from Equation (7) together with the constraint Equation (1) and discuss their properties. In Section 3 we shall briefly discuss the equivalence of our nonlinear formulation, upon quantization, with the usual quantum electrodynamics. In Section 4 the relevance of the above nonlinear choice of gauge for the emission of soft photons is briefly mentioned, and lastly in Section 5 our results are summarized and discussed.

## 2. SOLUTIONS AND PROPERTIES

We begin this section by first exhibiting the equations of motion obtained from the Lagrangian density Equation (7) when the electromagnetic potential $A_\mu$ satisfies the constraint Equation (1). Because of the constraint, the independent dynamical variables are $A_b$ and of course

$A_0 = (\rho^2 + A_b^2)^{1/2}$, therefore the equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial A_b} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_b} = \partial_\alpha F_{\alpha b} - i \frac{A_b}{A_0} \partial_a F_{4a} = 0 \tag{8}$$

Let us now examine the solutions to the above nonlinear equations.

A time-dependent solution can be obtained by setting (Nambu, 1968)[3]

$$A_0 = \left(\rho^2 + A_i^2\right)^{1/2} = \sqrt{\rho}\left(1 + A_i^2/2\rho^2 + \cdots\right) \tag{9}$$

where we assume $|A_i|$ ($\equiv \varepsilon$) is small and $\rho^2$ large. On keeping terms to lowest order in $\varepsilon$ we obtain the "free" equation of motion and solution ($A_b^F$):

$$\partial_\alpha^2 A_b^F - \partial_a \partial_b A_a^F = 0 \tag{10}$$

$A_b^F$ may be separated into transverse and longitudinal parts and it is straightforward to verify that it is the transverse part that behaves like the normal light wave whereas the longitudinal part is time independent.[4]

A static solution satisfying Equation (1) is given by (Righi and Venturi, 1978a)

$$A_i(\mathbf{x}) = \sqrt{\rho}\,\frac{x_i}{r}\sinh g(r) \equiv A_i^c(\mathbf{x})$$

$$A_4(\mathbf{x}) = \pm i\sqrt{\rho}\cosh g(r) \equiv \pm A_4^c(\mathbf{x}) \tag{11}$$

with $r = (x_i^2)^{1/2}$ and

$$\cosh g(r) = 1 \qquad \text{for } r < r_0$$

$$= r_0/r \qquad \text{for } r > r_0 \tag{12}$$

where $r_0$ satisfies $g(r_0) = 0$ and[5]

$$\partial_a F_{a4}^c = -\frac{1}{r_0}\delta(r - r_0) A_4^c \tag{13}$$

with $F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c$.

[3]In particular Nambu was interested in considering the photon as the massless excitation arising as a result of the spontaneous breakdown of Lorentz symmetry in analogy with the pion in the context of chiral symmetry.

[4]That is, the transverse solution $A_T^F$ satisfies $\Box A_T^F = 0$ whereas for the longitudinal one $A_L^F$ we have $\partial_0^2 A_L^F = 0$.

[5]An equivalent alternative way to our solutions is to consider the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}k(A_\mu^2 + \rho)$, where $k(x)$ may be treated as a further field variable. The equations of motion then are $\partial_\mu F_{\mu\nu} = kA_\nu$ together with Equation (1). The elimination of $k$ from the simultaneous equations of motion for $\nu = n$ and $\nu = 4$ will of course lead to the nonlinear form Equation (8).

We observe that with the above ansatz Equation (11) one has $F^c_{ij}=0$ and

$$F^c_{4i}=i\sqrt{\rho}\,\frac{x_i r_0}{r^3}\,\theta(r-r_0)$$

corresponding to the fields associated with a "charged shell" of radius $r_0$ (zero magnetic and nonzero electric fields). Further if we identify the electric charge $e$ ($>0$):

$$e/4\pi \equiv \sqrt{\rho}\, r_0 \tag{14}$$

and further set (see Section 1) $\sqrt{\rho}=m/e$,[6] we obtain

$$r_0=e^2/4\pi m \tag{15}$$

which is the classical charged particle radius.

Before examining in more detail the properties of our static extended solution let us observe that although it destroys gauge invariance our constraint leads to the existence of a current[7]

$$N_\mu=\frac{i}{6\pi^2}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma}A_\alpha\partial_\nu A_\beta\partial_\rho A_\gamma\partial_\sigma A_\delta \tag{16}$$

which is conserved:

$$\frac{\partial N_\mu}{\partial x_\mu}=0 \tag{17}$$

since the Jacobian of the four fields $A_\mu$ with respect to the four space-time

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Clearly $kA_\nu$ is a conserved current and leads to ($k\neq 0$) a conserved "charge." The introduction of $k(\neq 0)$ then implicitly means that we are introducing a dimensionless (for $\hbar=c=1$) constant which we may identify with the electric charge. The case $k=0$ of course corresponds to our solution $A^F$. In this case our constraint Equation (1) is just a choice of gauge in a gauge-invariant equation.

[6] Let us note that with such an identification the trace of energy-momentum tensor is equal to $m$.

[7] The normalization can be suitably chosen. We loosely refer to Eq. (16) as a topological current since unlike a Noether current it is not associated with a symmetry of the Lagrangian and is conserved by construction and independently of the equations of motion. This conserved quantity appears because of the manifold ($A_\mu^2=-\rho^2$) on which the field variables ($A_b$) take their values. In particular if, for simplicity, we consider Euclidean rather than Minkowski four space our constraint becomes $A_0^2+A_b^2=\rho^2$ which defines a field manifold $S^3$ and a finite energy static solution is a mapping of $R^3\cup\{\infty\}$ into the manifold. A current such as Eq. (16) is related to such a mapping.

coordinates must vanish (they are functionally dependent). The corresponding conserved charge is

$$N=\frac{1}{6\pi^2}\int d^3x\,\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{4nrs}A_\alpha\partial_n A_\beta\partial_r A_\gamma\partial_s A_\delta \tag{18}$$

which on substituting our ansatz, Equation (11), and solution leads to

$$N=\pm 4\frac{i\rho^4}{\pi}\int dr\, g'\sinh^2 g=\pm\rho^4 \tag{19}$$

the positive or negative signs in the above corresponding to the sign of our solution. Further our time-dependent solution is associated with $N=0$. Thus our conserved topological charge is proportional to the charged particle number and further since this is proportional to the charge it may be that charge conservation is connected to a topological conservation law.

Let us now examine the stability[8] of our static spherically symmetric solution $A_\mu^c$. The energy-momentum tensor is given by

$$T_{\mu\nu}=-\frac{1}{4}F_{\alpha\beta}^2\delta_{\mu\nu}-\left(-F_{\mu b}+F_{4\mu}\frac{iA_b}{A_0}\right)(F_{\nu b}+\partial_b A_\nu) \tag{20}$$

and the Hamiltonian $H$ is

$$H=\int d^3x\,T_{00}=\int d^3x\left[\frac{1}{2}(\mathbf{B}^2+\mathbf{E}^2)-A_0\nabla\cdot\mathbf{E}\right] \tag{21}$$

where $F_{cd}\equiv B_b\varepsilon_{bcd}$, $F_{a4}\equiv -iE_a$ and $\mathbf{E}$ and $\mathbf{B}$ are of course the electric and magnetic fields.

On substituting our ansatz Equation (11) into $H$ we obtain

$$H=4\pi\rho^2\int r^2\,dr\left[\frac{1}{2}\left(\frac{d}{dr}\cosh g\right)^2-\frac{\delta(r-r_0)}{r_0}\cosh^2 g\right] \tag{22}$$

and for a variation $\delta(\cosh g)$

$$\delta H=4\pi\rho^2\int dr\left[2r\frac{d}{dr}\cosh g+r^2\frac{d^2}{dr^2}\cosh g-\frac{r^2}{r_0}\delta(r-r_0)\cosh g\right]\delta(\cosh g) \tag{23}$$

[8]Let us observe that because of the structure of $\mathfrak{L}$ the usual instability theorems based on scaling are evaded even without adding to $\mathfrak{L}$ quartic terms such as $-(\varepsilon^2/8)[(F_{\mu\nu}F_{\mu\nu})^2-2(F_{\mu\alpha}F_{\alpha\nu})^2]$ which give a positive definite contribution to the Hamiltonian.

which is zero in virtue of the equations of motion. The second order variation is given by

$$\delta^2 H = 4\pi\rho^2 \int r^2\, dr \left\{ \frac{1}{2}\left[\delta\left(\frac{d}{dr}\cosh g\right)\right]^2 + \frac{\delta(r-r_0)}{r_0}\left[\delta(\cosh g)\right]^2 \right\} \quad (24)$$

which is positive. Thus our solution is stable with respect to static spherically symmetric perturbations (Righi and Venturi, 1978b).[9]

The stability of our solution with respect to general time-dependent infinitesimal fluctuations (Righi and Venturi, 1978a) may be checked by solving for the eigenfrequencies of the small perturbations, the solution then being stable or not depending on whether the eigenfrequencies are real or complex. Let us then set

$$A_\mu = A_\mu^c + a_\mu \quad (25)$$

where $a_\mu$ is the small perturbation and $A_\mu^c$ is given by[10] Equations (11) and (12).

Upon substituting into Equations (1) and (8) and just keeping terms to lowest order in the perturbation we obtain

$$a_0 \cosh g = \frac{a_i x_i}{r} \sinh g \quad (26)$$

and

$$\partial_\alpha f_{\alpha b} + \frac{i x_b}{r}(\tanh g)\partial_a f_{a4} = -\frac{a_b}{r}\delta(r-r_0) \quad (27)$$

respectively, where $f_{\alpha\beta} \equiv \partial_\alpha a_\beta - \partial_\beta a_\alpha$. On taking the divergence of both sides of Equation (27) we also get

$$\partial_b \partial_4 f_{4b} + \partial_b\left[\frac{i x_b}{r}(\tanh g)\partial_a f_{a4}\right] = -\partial_b\left[\frac{a_b}{r}\delta(r-r_0)\right] \quad (28)$$

which is the equivalent of the current conservation condition.

Since one is interested in whether the perturbation, which is required to be well behaved (small) in all space, remains small or increases in time, the time dependence of a given vibrational mode is represented by an exponen-

[9]It is further straightforward to check from Equation (20) and our spherical solution that the gradient of the stress tensor in our case is zero in contrast with the case of the Lorentz electron.

[10]We shall henceforth consider the solution with the plus sign in the latter of Equations (11).

tial. Further, since our solution is spherically symmetric, it is convenient to expand **a** in terms of vector spherical harmonics $\mathbf{Y}_{j,l,m}$. One then sets

$$\mathbf{a}=e^{i\omega t}\sum_{l} F_l(r)\mathbf{Y}_{j,l,m}(\hat{r}) \tag{29}$$

where in the sum $l$ takes the values $l=j+1,\ j,\ j-1$.

On substituting for **a** into Equations (27) and (28) and using Equation (26) we obtain the following:

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}j(j+1)\right]F_0+\omega^2F_0=-\frac{\delta(r-r_0)}{r}F_0 \tag{30}$$

$$\begin{aligned}
&\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}(j+1)(j+2)\right]F_+-\alpha_+\left(\frac{d}{dr}-\frac{j}{r}\right)\\
&\quad\times\left[\sum_{\pm}\frac{dF_\pm}{dr}\alpha_\pm+F_+\alpha_+\frac{(j+2)}{r}-F_-\alpha_-\frac{(j-1)}{r}\right]+\omega^2F_+\\
&\quad-i\omega\alpha_+\left(\frac{d}{dr}-\frac{j}{r}\right)\sum_{\pm}\alpha_\pm(\tanh g)F_\pm-\alpha_+(\tanh g)\\
&\quad\times\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}j(j+1)\right]\sum_{\pm}\alpha_\pm(\tanh g)F_\pm-i\omega\alpha_+(\tanh g)\\
&\quad\times\left[\sum_{\pm}\frac{dF_\pm}{dr}\alpha_\pm+F_+\alpha_+\frac{(j+2)}{r}-F_-\alpha_-\frac{(j-1)}{r}\right]=-\frac{\delta(r-r_0)}{r}F_+
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}(j-1)j\right]F_--\alpha_-\left[\frac{d}{dr}+\frac{(j+1)}{r}\right]\\
&\quad\times\left[\sum_{\pm}\frac{dF_\pm}{dr}\alpha_\pm+F_+\alpha_+\frac{(j+2)}{r}-F_-\alpha_-\frac{(j-1)}{r}\right]+\omega^2F_-\\
&\quad-i\omega\alpha_-\left[\frac{d}{dr}+\frac{(j+1)}{r}\right]\sum_{\pm}\alpha_\pm(\tanh g)F_\pm-\alpha_-(\tanh g)\\
&\quad\times\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}j(j+1)\right]\sum_{\pm}\alpha_\pm(\tanh g)F_\pm-i\omega\alpha_-(\tanh g)\\
&\quad\times\left[\sum_{\pm}\frac{dF_\pm}{dr}\alpha_\pm+F_+\alpha_+\frac{(j+2)}{r}-F_-\alpha_-\frac{(j-1)}{r}\right]=-\frac{\delta(r-r_0)}{r}F_-
\end{aligned} \tag{32}$$

$$\left[\frac{2}{r}(\tanh g)+\frac{d}{dr}(\tanh g)+(\tanh g)\frac{d}{dr}+i\omega\right]$$

$$\times\left\{\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{1}{r^2}j(j+1)\right]\sum_{\pm}\alpha_{\pm}(\tanh g)F_{\pm}\right.$$

$$\left.-i\omega\left[\sum_{\pm}\alpha_{\pm}\frac{dF_{\pm}}{dr}+F_{+}\alpha_{+}\frac{(j+2)}{r}-F_{-}\alpha_{-}\frac{(j-1)}{r}\right]\right\}$$

$$=\left\{\sum_{\pm}\alpha_{\pm}\frac{d}{dr}\left[F_{\pm}\frac{\delta(r-r_0)}{r}\right]+F_{+}\frac{\delta(r-r_0)}{r}\alpha_{+}\frac{(j+2)}{r}\right.$$

$$\left.-F_{-}\frac{\delta(r-r_0)}{r}\alpha_{-}\frac{(j-1)}{r}\right\} \tag{33}$$

where

$$\alpha_{+}=-\left(\frac{j+1}{2j+1}\right)^{1/2}, \qquad \alpha_{-}=\left(\frac{j}{2j+1}\right)^{1/2}$$

and

$$F_{j\pm1}\equiv F_{\pm}, \qquad F_j\equiv F_0$$

Bearing in mind that the solutions for **a** must be less than $\mathbf{A}^c$ for all $r$ we first obtain the solution for the small fluctuations inside the "shell" ($r<r_0$) and then observe that it will vary continuously from the solution for $r<r_0$ to that for $r>r_0$, whereas its derivative may be discontinuous. From Equations (30)–(33) we then get that

$$F_0(r)=-F_0(r_0)r_0\omega j_j(\omega r_<)n_j(\omega r_>) \tag{34}$$

where $j_j$ and $n_j$ are the spherical Bessel and Neumann functions, respectively, and $r_<$ ($r_>$) is the lesser (greater) of $r$ and $r_0$. It is then immediate to see that the above solution, which leads to an eigenvalue condition for $\omega$, excludes complex values of $\omega$; otherwise it would "explode" as $r\to\infty$.

Further we have for all $r$

$$F_{+}=F_{-}=0 \tag{35}$$

which implies that fluctuations of the above type are not compatible with the constraints and the equations of motion.

We may then deduce that small, time-dependent perturbations around our classical solution are either not allowed or are associated with real eigenfrequencies, thus implying its stability. We feel such an analysis of stability to be equivalent to the search for the so-called "run-away" solutions in classical electrodynamics, stability implying their absence.

As we have noted, our solution is associated with a zero magnetic and a nonzero electric fields. Since a static magnetic moment (spin) is expected to be a small correction $[O(\hbar)]$ to our solution, let us modify our ansatz Equation (11) so as to allow for a nonzero magnetic field:

$$A_i(\mathbf{x}) = \sqrt{\rho}\,\frac{x_i}{r}\sinh g(r) + \frac{1}{2}\varepsilon_{ijk} d_j x_k \hbar f(r) \equiv A_i'^c(\mathbf{x}) \tag{36}$$

leaving $A_4^c$ unchanged. In the above we have introduced another direction, besides that of $\mathbf{x}$, which is represented by some unit axial vector $\mathbf{d}$ which will be associated with the spin orientation. We further observe that the above ansatz satisfies our constraint to $O(\hbar^2)$.

On substituting Equation (36) into our equations of motion and keeping terms to lowest order in $\hbar$ we get Equation (27) with $a_\mu$ replaced by $\varepsilon_{ijk} d_j x_k \hbar f(r)$, which leads to

$$\frac{d^2 f}{dr^2} + \frac{4}{r}\frac{df}{dr} = \frac{-f(r_0)}{r_0}\delta(r - r_0) \tag{37}$$

and for $r > r_0$ a solution to the above is given by[11]

$$F(r) = \frac{1}{4\pi\sqrt{\rho}\, r^3} \tag{38}$$

which on setting $\sqrt{\rho} = m/e$ corresponds to the field of a magnetic dipole $(e\hbar/2m)\mathbf{d}$. Some motivation can also be given for associating a gyromagnetic ratio $g$ of 2 with the above solution (Righi and Venturi, 1979a).

As we have seen, our nonlinear equations of motion have two solutions: a time-dependent one associated with "small" (with respect to $\sqrt{\rho}$) fields corresponding to photons, and a "large" static one associated with charged particles (electrons). Further, on treating a magnetic dipole moment as a first-order quantum effect, we find that it also can arise as a solution to our equations of motion. Because of our constraint we also have a conserved topological current whose associated conserved charge is proportional to the charged particle number. Lastly we remember the stability of our extended-particle-like solution with respect to small perturbations.

[11] Naturally $f$ multiplied by a constant is also a solution to Equation (37).

## 3. QUANTIZATION AND CHARGED FIELD OPERATORS

Having exhibited our solutions and their properties we may now proceed by quantizing and constructing the charged-field operators. Because of our constraint Equation (1) the independent dynamical variables are the $A_i$ with conjugate momenta $iF_{4i}$ and satisfy the canonical equal time commutation relations

$$[A_i(\mathbf{x},t), F_{4j}(\mathbf{y},t)] = \delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \tag{39}$$

which leads to the following commutator for the free-field solutions $A_i^F = A_i^{F(+)} + A_i^{F(-)}$:

$$\begin{aligned}[A_i^{F(+)}(x), A_j^{F(-)}(y)] &= \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial_k^2}\right)\frac{1}{(2\pi)^3}\int\frac{d^3k}{2k_0}e^{ik\cdot(x-y)} \\ &= \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial_k^2}\right) i\Delta^{(+)}(x-y)\end{aligned} \tag{40}$$

where $k_0^2 = \mathbf{k}^2$. We remark that it is only the transverse part[4] of the free field that is quantized and that the other free-field commutation relations can be obtained from Equation (40). We may now discuss the construction of the charged-particle operator (Righi and Venturi, 1979b). An operator $\varphi$ which creates a charged particle as given by Equation (11)[10] must have the following properties:

$$[A_i(\mathbf{y},t), \varphi(\mathbf{x},t)] = \varphi(\mathbf{x},t)A_i^c(\mathbf{y}-\mathbf{x}) \tag{41}$$

$$[F_{4i}(\mathbf{y},t), \varphi(\mathbf{x},t)] = \varphi(\mathbf{x},t)F_{4i}^c(\mathbf{y}-\mathbf{x}) \tag{42}$$

and we observe that the right-hand side of the above equations vanishes for $|\mathbf{y}-\mathbf{x}| < r_0$. We shall generally consider $|\mathbf{y}-\mathbf{x}| \gg r_0$, or equivalently $r_0 \to 0$.

A suitable expression for $\varphi$ satisfying the above commutation relations is given by

$$\begin{aligned}\varphi(\mathbf{x},t) &= \lim_{r_0\to 0} c\sqrt{\rho}\, :\exp\left\{\left(1 + r_0^2\partial_\nu^2\right)\int d^3\eta\left[F_{4j}(\boldsymbol{\eta}+\mathbf{x},t)A_j^c(\boldsymbol{\eta})\right.\right. \\ &\qquad \left.\left. - A_j(\boldsymbol{\eta}+\mathbf{x},t)F_{4j}^c(\boldsymbol{\eta})\right]\right\}: \\ &= \lim_{r_0\to 0} c\sqrt{\rho}\, :\exp\left[-i\left(1 + r_0^2\partial_\nu^2\right)\int_\sigma d\sigma^\mu\left(F_{\mu\nu}A_\nu^c - A_\nu F_{\mu\nu}^c\right)\right]:\end{aligned} \tag{43}$$

where $c$ is a constant we shall determine and $\sigma$ and $d\sigma^\mu$ are a general spacelike surface and element of surface, respectively.

Having now constructed the charged particle fields we may obtain the electromagnetic current density, $j_\mu$, which we assume is given by

$$j_\mu(x) = -i\varphi^\dagger(x)\overleftrightarrow{\partial}_\mu\varphi(x) \tag{44}$$

We evaluate it by using Equation (43) and

$$\partial_\mu :e^A: = \partial_\mu A :e^A: - \left[\partial_\mu A^{(+)}, A^{(-)}\right] :e^A: \tag{45}$$

$$:e^A::e^B: = e^{[A^{(+)}, B^{(-)}]} :e^{A+B}: \tag{46}$$

which are true if $[\partial_\mu A^{(+)}, A^{(-)}]$ and $[A^{(+)}, B^{(-)}]$ are $c$ numbers. We shall use Equation (40) whenever a commutator between positive and negative frequency terms is necessary and keep terms to lowest order in $\varepsilon$ [see Equation (9) for example] only. Further, whenever necessary the integrals we encounter in our determination of the current are approximately evaluated by setting[12]

$$A_i^c = i\sqrt{\rho}\,\frac{x_i}{r}\theta(r - r_0)$$

$$A_4^c = i\sqrt{\rho}\left[1 - \theta(r - r_0)\right] \tag{47}$$

with $r_0 \to 0$. This implies that we are considering the point limit of our classical solution.

According to the above considerations for example we find

$$\begin{aligned}\varphi^\dagger(x)\overleftrightarrow{\partial}_4\varphi(x) &\approx -2c^2\rho^2\int d^3\eta\,\partial_4 A_j(\boldsymbol{\eta} + \mathbf{x}, t)F_{4j}^c(\boldsymbol{\eta}) \\ &= -2c^2\rho\int d^3\eta\,\partial_j F_{4j}(\boldsymbol{\eta} + \mathbf{x}, t)A_4^c(\boldsymbol{\eta}) \\ &\approx -i\partial_j F_{4j}(x)c^2\rho^3\tfrac{8}{3}\pi r_0^3\end{aligned} \tag{48}$$

and similarly

$$j_\mu(x) \approx c^2\rho^3\frac{8\pi}{3}r_0^3\partial_\nu F_{\nu\mu}(x) \equiv \frac{1}{e}\partial_\nu F_{\nu\mu}(x) \tag{49}$$

[12]As we have mentioned we shall take the solution with positive sign for $A_4$. Further we observe that $A_i^c$ is imaginary, as is necessary to maintain our constraint Equation (1). This is not a difficulty since the observable quantities are the $F_{\mu\nu}^c$ which are real.

which allows us to set

$$c^2 = \frac{3}{8\pi e}(r_0\rho)^{-3} = \frac{24\pi^2}{e^4} \tag{50}$$

Let us observe that, as expected with our constraint [Equation (1)], the current is associated with the longitudinal degrees of freedom for the electromagnetic field.

With the above expression for the current we can obtain the commutator

$$[j_4(\mathbf{x},t),\varphi(\mathbf{y},t)] = \left[\frac{1}{e}\partial_i F_{i4}(\mathbf{x},t),\varphi(\mathbf{y},t)\right] = -\frac{4\pi r_0\sqrt{\rho}}{e} i\delta^3(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y},t) \tag{51}$$

which on using Equation (14) is the expected result for a charged field. Further on using

$$:e^A::e^B: = e^{[A,B]}:e^B::e^A: \tag{52}$$

where $[A, B]$ is a $c$ number we can easily verify that

$$\begin{aligned} &[\varphi(\mathbf{x},t),\varphi(\mathbf{y},t)] = 0 \\ &[\varphi^\dagger(\mathbf{x},t),\varphi(\mathbf{y},t)] = 0 \end{aligned} \tag{53}$$

which are the expected commutators for Bose operators. We now observe that we have the expected canonical commutation relations for the charged scalar field $\varphi$ since the remaining commutator between $\varphi$ and $\dot{\varphi}^\dagger (\equiv \partial_0\varphi^\dagger)$

$$[\varphi(\mathbf{y},t),\dot{\varphi}^\dagger(\mathbf{x},t)] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{54}$$

is implied by Equation (51).

The same procedure and approximations may also be employed to determine the equations of motion for the scalar field, obtaining

$$\begin{aligned} &\partial_\mu^2\varphi(x) \\ &\quad\approx (1 + r_0^2\partial_\nu^2)c\rho\int d^3\eta\left[\partial_\mu^2 F_{4j}(\boldsymbol{\eta}+\mathbf{x},t)A_j^c(\boldsymbol{\eta}) - \partial_\mu^2 A_j(\boldsymbol{\eta}+\mathbf{x},t)F_{4j}^c(\boldsymbol{\eta})\right] \\ &\quad\approx 0 \end{aligned} \tag{55}$$

which is compatible with Equation (49) and current conservation.

In order to understand the above results it is instructive to compare them with the equations of motion obtained from the usual Lagrangian density $\mathcal{L}'$, Equation (2), for a charged scalar field interacting with the electromagnetic field on employing the constraint Equation (1) and making the usual approximation ($\varepsilon$ small). For the equation of the electromagnetic field one obtains

$$A_4(\partial_\alpha F_{\alpha i} - ej_i) = A_i(\partial_\alpha F_{\alpha 4} - ej_4) \tag{56}$$

which on comparing terms of $O(\varepsilon)$ and $O(\varepsilon^2)$ leads to

$$\begin{aligned} \partial_\alpha F_{\alpha i} &= ej_i \\ \partial_\alpha F_{\alpha 4} &= ej_4 \end{aligned} \tag{57}$$

in agreement with Equation (49). For the scalar field the equation of motion to lowest order in $\varepsilon$ is given by

$$\partial_\mu^2 \varphi \approx 0 \tag{58}$$

in agreement with Equation (55), and we note that the eventual presence of constant potential terms acting on the charged fields need not be considered since they can be eliminated through a redefinition of the charged particle momentum.

In the previous section we have shown that a magnetic dipole moment solution arises naturally as a small $[O(\hbar)]$ perturbation to the charged particle solution of our equations of motion. The existence of such a solution is closely associated with the introduction of an additional direction in space, which is expected to be related to the spin orientation. Let us now illustrate how the presence of this additional direction in space allows us to construct a field satisfying anticommutation relations (Righi and Venturi, 1979a).

An operator $\psi$ which creates a charged particle as given by Equation (35) must have the following property:

$$[A_i(\mathbf{y}, t), \psi(\mathbf{x}, t)] = \psi(\mathbf{x}, t) A_i'^c(\mathbf{y} - \mathbf{x}) \tag{59}$$

for $|\mathbf{y} - \mathbf{x}| \gg r_0$ (or equivalently $r_0 \to 0$) and a suitable representation for $\psi$ is given by

$$\psi(\mathbf{x}, t) = \lim_{r_0 \to 0} \left( \frac{48\pi^2}{e^3} \right)^{1/2} \rho^{3/2} \colon \exp\left\{ \frac{\pi}{2\sqrt{\rho}} \mathbf{d} \cdot \mathbf{A} + (1 + r_0^2 \partial_\nu^2) \int d^3\eta \right.$$
$$\left. \times \left[ F_{4j}(\boldsymbol{\eta} + \mathbf{x}, t) A_j^c(\boldsymbol{\eta}) - A_j(\boldsymbol{\eta} + \mathbf{x}, t) F_{4j}^c(\boldsymbol{\eta}) \right] \right\} \colon \tag{60}$$

With respect to the scalar case we note the presence of the additional pseudoscalar term $(\pi/2\sqrt{\rho})\mathbf{d}\cdot\mathbf{A}$ in the exponent and observe that since the actual construction and identification of the different spin components is difficult and ambiguous, we shall just briefly mention the properties of the above simple solution.

On using Equation (52) we obtain for $r \gg r_0$

$$\psi(\mathbf{x},t)\psi(\mathbf{y},t)=\exp i\pi\mathbf{d}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\psi(\mathbf{y},t)\psi(\mathbf{x},t) \tag{61}$$

which on taking the unit axial vector $\mathbf{d}$ either parallel or antiparallel to $(\mathbf{x}-\mathbf{y})$ (which is equivalent to a quantization of spin) leads to

$$\{\psi(\mathbf{x},t),\psi(\mathbf{y},t)\}=0 \tag{62}$$

which is the expected anticommutation relation. The equations satisfied by $\psi$ may be obtained in a manner analogous to that employed for the scalar field case and the results are essentially the same (Righi and Venturi, 1979a). This is because on considering the point particle and weak field limits we expect $\rho\to\infty$ and $\varepsilon\to 0$ and therefore the additional term in the exponent of $\psi$ [see equation (60)] will not contribute unless compensations occur, as in the case for the anticommutator Equation (61); thus a detailed examination of spin effects is actually precluded.

Let us, however, end this section by mentioning that one can examine directly quantum fluctuations about the charged magnetic dipole solution to our nonlinear equations, thus bypassing the actual construction of the spinor fields. The effective Hamiltonian density describing the interaction between the small quantum fluctuations (photons) and our charged-particle solution and the photon Feynman propagator in our "gauge" can be constructed. The lowest-order effect on the magnetic moment due to the emission and the reabsorption of a photon can then be evaluated in the point particle limit and the usual value $(\alpha/2\pi)$ for $(g-2)/2$ is then recovered (Righi and Venturi, 1979c).

## 4. QUANTUM ELECTRODYNAMICS IN A NONLINEAR GAUGE

In the previous section we have given arguments for the quantum mechanical equivalence of our approach with the usual quantum electrodynamics. We may now regard the latter as a theory in which the charged-particle solutions and their associated field operators have been separated from the electromagnetic field and corresponding to a sort of "effective Lagrangian" approach. Let us now examine the consequences of our constraint Equation (1) in such a context.

In terms of quantized fields the constraint Equation (1) implies that the electromagnetic field has a nonzero vacuum expectation value, this implying the spontaneous breaking of Lorentz symmetry. However, since for the free electromagnetic field Equation (1) is just a choice of gauge (Dirac, 1951, 1952, 1954; Nambu, 1968)[5] in a perturbative approach the $S$-matrix elements will just be the usual gauge-invariant and Lorentz-invariant ones (Nambu, 1968).

We observe that in general the gauge is chosen opportunely depending on the process one wishes to consider. Let us therefore exhibit the interest of the choice Equation (1) for the emission of soft photons (Righi and Venturi, 1977). The usual Lagrangian density describing photons $A_\mu$, electrons $\psi$, and their interaction is given by

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 - \bar{\psi}\gamma_\mu\partial_\mu\psi - m\bar{\psi}\psi + ie\bar{\psi}\gamma_\mu A_\mu\psi \tag{63}$$

and since the electromagnetic field may be suitably redefined without altering the $S$ matrix (Chisholm, 1961) we may set

$$\begin{aligned} A_0 &= \pm\left[\rho^2 + \boldsymbol{\varphi}^2F^2(\boldsymbol{\varphi}^2)\right]^{1/2} \\ A_i &= \varphi_i F(\boldsymbol{\varphi}^2) \end{aligned} \tag{64}$$

where $F(0)=1$ and the above clearly satisfies Equation (1). In particular let us choose[13]

$$F(\boldsymbol{\varphi}^2) = \sqrt{\rho}\,\frac{\sinh(\boldsymbol{\varphi}^2/\rho^2)^{1/2}}{(\boldsymbol{\varphi}^2)^{1/2}} \tag{65}$$

and with the above choice the interaction Lagrangian density becomes

$$\begin{aligned} \mathcal{L}_{\text{int.}} &= ie\bar{\psi}\gamma_\nu A_\nu\psi = \pm e\sqrt{\rho}\,\bar{\psi}\gamma_4\exp\left(\pm\frac{i}{\sqrt{\rho}}\gamma_4\boldsymbol{\varphi}\cdot\boldsymbol{\gamma}\right)\psi \\ &= \pm e\sqrt{\lambda}\bar{\psi}\exp\left(\mp\frac{i}{2\sqrt{\rho}}\gamma_4\boldsymbol{\varphi}\cdot\boldsymbol{\gamma}\right)\gamma_4\exp\left(\pm\frac{i}{2\sqrt{\rho}}\gamma_4\boldsymbol{\varphi}\cdot\boldsymbol{\gamma}\right)\psi \end{aligned} \tag{66}$$

which is a possible form for our nonlinear interaction Lagrangian and we

[13] As we shall see, this choice leads naturally to the use of coherent states for photons.

observe the natural emergence of coherent states for photons. For simplicity we shall henceforth consider the solution with the + sign in Equations (64) and (66).

On examining Equation (66) we see that the coefficient of the spinor $\psi$ has the structure of a Lorentz boost depending on the electromagnetic field $\boldsymbol{\varphi}$. The effect of this coefficient is such that under a Lorentz transformation the field

$$\hat{\psi} \equiv \exp\left( \frac{i}{2\sqrt{\rho}} \gamma_4 \boldsymbol{\varphi} \cdot \boldsymbol{\gamma} \right) \psi \tag{67}$$

just undergoes a rotation and therefore the right-hand side of Equation (66) is Lorentz invariant.

The above observation allows us to study another application of the above approach to the emission of soft photons in analogy with the emission of soft pions (Weinberg, 1970a, b; Brown, 1970). Let us consider a process involving the scattering of a charged particle $\psi$ and describe it by an effective Lagrangian density

$$\mathcal{L}_{\text{eff.}} = \bar{\psi}(x) T_0 \psi(x) \tag{68}$$

which reproduces the basic collision process in first Born approximation. Since the photon is associated with the spontaneous breaking of Lorentz symmetry we shall just assume rotational invariance for the above effective Lagrangian in the absence of electromagnetic interactions.

The electromagnetic interaction may then be introduced by coupling photons in the effective Lagrangian Equation (68) so as to render it Lorentz invariant. This can be done by replacing $\psi$ by $\hat{\psi}$. That is, the $S$ matrix for our basic collision process becomes (omitting inessential factors)

$$S = i \int d^4x \exp(-iq \cdot x) \bar{u}(p+q) \exp\left[ -\frac{i}{2\sqrt{\rho}} \gamma_4 \boldsymbol{\varphi}(x) \cdot \boldsymbol{\gamma} \right] T_0 \times \exp\left[ \frac{i}{2\sqrt{\rho}} \gamma_4 \boldsymbol{\varphi}(x) \cdot \boldsymbol{\gamma} \right] u(p) \tag{69}$$

where $q$ is the momentum transfer in the basic process. The above of course is for collisions involving a charged spin-1/2 object. It is straightforward to consider either other spin particles or the presence of more charged particles. We again observe that our results have no infrared divergences and depend on the vacuum expectation value of the electromagnetic field $\sqrt{\rho}$ instead of the charge (Weinberg, 1970a, b; Brown, 1970).

If we take the vacuum expectation value of the above and employ translational invariance we get the correction to the basic collision process due to the exchange of virtual soft photons, which we may write schematically as

$$S = \left\langle 0 \left| \exp\left[ \frac{i}{2\sqrt{\rho}} \varphi(0)\Delta(\gamma_4\gamma) \right] \right| 0 \right\rangle S_0 \tag{70}$$

The actual evaluation of the above presents some difficulties (Weinberg, 1970a, b; Brown, 1970).

## 5. CONCLUSIONS

In the Introduction we have exhibited the motivation for the nonlinear constraint on the electromagnetic field given by Equation (1). Essentially the advantage of such a choice of constraint (or "gauge") is that the unphysical degrees of freedom for the electromagnetic field, which would otherwise be eliminated because of gauge invariance, are used so as to describe the charged field (particle). Such a constraint is possible for the electromagnetic field to the extent that there exists a solution to the corresponding classical equations of motion for a charged particle in that field (Dirac, 1951, 1952, 1954; Nambu, 1968).

As a starting point we then implemented the usual gauge-invariant Lagrangian density for the electromagnetic field, Equation (7), with the constraint Equation (1) and examined the solutions to the resulting nonlinear equations of motion. We found two types of solutions: a time-dependent one associated with "small" (with respect to $\sqrt{\rho}$) fields corresponding to photons and a "large" static spherically symmetric one associated with charged particles.

The static solution resembles a spherical charged shell whose radius is just the classical radius of a charged particle. Further, our constraint, although destroying gauge invariance, leads to the existence of a topological, trivially conserved, current and charge. We find that our static solution has a nonzero topological charge which is proportional to the charged-particle number, that is the charge, thus indicating that charge conservation may be connected to a topological conservation law.

On then checking the stability of our extended solution, we first find it to be stable for time-independent spherically symmetric perturbations. General time-dependent infinitesimal fluctuations about our solution were then examined. For such a case the stability criterion is that infinitesimal fluctuations which are well behaved (small) in all space are not associated with complex eigenfrequencies, otherwise they would "explode" in time. We found, by examining the continuity of the solution in a singularity-free space (inside the "shell") to the solution in the presence of a charge associated potential (outside the "shell"), that the perturbation is either not allowed or leads to real eigenfrequencies implying that the solution is stable. We feel such an analysis of stability to be equivalent to the search for the so-called "run-away" solutions in classical electrodynamics, stability implying their absence. Concerning small perturbations to our solution we also observed that a magnetic dipole moment can easily be accommodated as a first-order quantum effect. The existence of such a magnetic dipole solution is closely associated with the introduction of an additional direction in space, which is expected to be related to the spin orientation.

In order to study the equivalence of our approach, once quantized, with the usual quantum electrodynamics, we observe that according to our approach the charged particles are also solutions to our equations of motion. Therefore there is no need to introduce new fields; rather the problem is to separate and discuss the interaction between "small" oscillations about a vacuum, satisfying linear equations, and static nonperturbative solutions satisfying the full nonlinear equations (8).

We have then constructed the charged-particle field operators in terms of the classical static solution and quantized small oscillations. In an idealized point particle limit, employing a necessarily heuristic procedure, we have obtained the charged-field equation of motion and the lowest-order corrections to the field equations for the quantized electromagnetic field due to the presence of charge. We further verified that in general the charged scalar field satisfies the correct commutation relations and that our results are equivalent to the usual ones for the electromagnetic interaction and equation of motion for a charged scalar field.

Since the magnetic dipole moment solution to our equations of motion introduces an additional direction in space, we have illustrated how one can then also construct a field satisfying anticommutation relations without explicitly identifying the diverse spinor components. Quantum fluctuations about the charged magnetic dipole solution can also be examined directly and the lowest-order quantum corrections evaluated. Agreement with the usual quantum electrodynamical value for $g-2$ is obtained.

Another remarkable property of our constraint Equation (1) is that when it is used within the context of the usual quantum electrodynamics (or if we wish in a theory where the charged-particle solutions and their associated field operators have been separated from the electromagnetic field) the photon is associated with the spontaneous breaking of Lorentz symmetry. It is then found that the emission of soft photons does not exhibit any infrared divergences. We observe, however, that in a perturbation expansion the final results are the usual Lorentz-invariant ones since for the free electromagnetic field Equation (1) is just a gauge choice in an otherwise gauge-invariant theory.

The properties of our approach under conformal transformations may also be considered (Righi and Venturi, 1980). In particular one may replace $\sqrt{\rho}$ by a field and interpret such a field as the additional component of a five-vector electromagnetic field. Correspondingly Minkowski space acquires an extra space dimension which may be related to either the radius or some internal coordinate conjugate to the rest mass of our solution. One is then naturally led to a Lagrangian density which is manifestly invariant under a larger conformal group $PO(5,2)$. In this case $\sqrt{\rho}$ is interpreted as the vacuum expectation value, associated with the spontaneous breaking of the conformal group, of the additional component of the five-vector electromagnetic field. Such a group is of interest since it has led to a theoretical expression for the fine-structure constant which agrees with the present experimental value to within one part per million (Righi and Venturi, 1980; Wyler, 1969, 1971). Therefore because of the apparent topological origin of change, one of the points worth investigating is the topological structure of our solutions; this may be more easily formulated in Euclidean rather than Minkowski space.

To conclude, it appears that our approach may lead to a formulation of quantum electrodynamics which while retaining the positive results of the usual approach is not plagued by divergence difficulties.

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